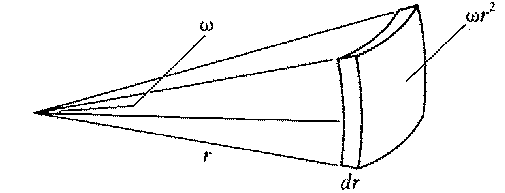
Distributions (Solutions) – Breakout Session on Mar 11, 2021

1. Uniform Distribution

Suppose the Sun is situated at the centre of a galaxy whose radius is exceedingly large. Furthermore, all the stars in the galaxy have the same absolute magnitude M, the stellar distribution has a uniform density *n* (stars per unit volume) and there is *no* interstellar attenuation.

Moreover, suppose you observe the sky in a small solid angle ω, as in the figure below:



*What would the apparent magnitude distribution function be in this situation, i.e., N(m), the number of stars with apparent magnitude m observed in this slice of sky?* [Hint: Begin with: *N*(r) = . Integrate. Now recall the relation between the apparent and absolute magnitudes as a function of distance, *r*. Solve for *r* as a function of m and M, and then finally *N* (or log(*N*)) as a function of m.] Now it turns out that observations of log(*N*) do not follow this functional form. *Propose reasons why not.*

If *N*(*r*) = (1/3) *n ω r*3 , then *r* = (3 *N*/ *n ω*)1/3 . Recall (m-M) = 5log(*r*/10) from which it can be shown that log(*r*) = 0.2m (1 – 0.2M). Notice that the quantity in brackets is constant for a particular class of star (e.g., G2V). (If stars are well mixed in the Galaxy, then this should be a constant in general, i.e., where there is a variety of stellar populations). Now since log(*N*) = 3log(*r*), then star counts (under the above assumptions) should show log(*N*) = 0.6m + constant. They do not in general. Why not? Because: a) of interstellar extinction in the plane (in which the Sun is situated), b) the Galaxy is neither finite nor spherical. (One can do the same thing with an extragalactic population such as radio sources or quasars or… In this case, deviations from 0.6m also can be from source evolution and the expanding universe.)

1. Binomial Distribution

Let’s imagine that the Hubble Space Telescope observed a sample of 52 globular star clusters. (This is not true, by the way!) The HST found that 11 of the 52 had “cuspy cores” (i.e., cores with a very high stellar density). The rest did not. Now the James Webb Space Telescope, Hubble’s successor, will observe a further sample of 137 other globular clusters in our Galaxy. *What is the probability that JWST will find 10, 20, 25 and 30 cuspy clusters in this new sample?*

This can be found via the binomial theorem where the probability of a “cuspy core” *p* = 11/52 or 0.212 (assuming the same fraction “success rate” will apply). For *N* clusters (137) with the number of “successes” *n*, then the probability of achieving n is given by:

P(*n*) = *N*! / ((*N-n*)! *n*!) *pn* (1-*p*)(N-n) . With *N*=137, and *n* = 10, 20, 25, 30, then:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *n* | 10 | 20 | 25 | 30 |
| *P* (percent) | < 0.001 | 1.4 | 6.1 | 8.0 |

NB: one has to be careful numerically here and use either a routine that keeps “double precision” or be clever in how one computes P. I strongly recommend using Ramunajan’s approximation for log(N!) where N is large. Amazing.

1. Poisson Distribution

One of the graduate students in this class took multiple, one-second images of a high-redshift galaxy through a particular filter. Upon reduction, the student found that the average photon flux recorded from the galaxy was 137 photons per second. *Considering only shot noise (and not detector noise), what is the probability that an exposure of one second would record 110, 120, 135, 145 and 155 photons?*

In a Poisson distribution, P(n) = *μn* e-*μ* / *n*! where *μ* is the average photon flux, and *n* is the anticipated count rate (where *μ*=137). [First taking the natural logarithm of each side is actually easier: ln(P(*n*)) = *n* ln(*μ*) – *μ* –ln(*n*!).] This leads to the percentage probability:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *N* | 110 | 120 | 135 | 145 | 155 |
| P(percent) | 0.2 | 1.2 | 3.4 | 2.6 | 1.0 |

1. Gaussian Distribution

While the mass or luminosity function for stars in general is a power law, it turns out that stars of a given spectral type, e.g., G2V (like our Sun), have an absolute magnitude distribution that is Gaussian in nature, with a mean absolute magnitude of +4.83 and a standard deviation of 0.11 magnitudes. *If the Sun has an absolute magnitude of +4.74, what is the probability that a G2V star randomly selected from a fair sample of such stars has a brightness equal to or greater than the Sun’s?*

The appropriate probability, P, is

The argument of the first error function is -0.579 (remember, brighter magnitudes are more negative), and so the probability that a G2V star is brighter than the Sun is 20.7%

1. Gaussian Distribution

For a particular culture, the height of adult males is normally distributed, with a mean of 175 cm and dispersion of 7.5 cm. Adult females’ mean height is 165 cm with the same dispersion. Suppose there are an equal number of males and females in this sample. What is the likelihood that a person 172 cm tall is female?

Here *μ*f = 165 cm, *μ*m = 175 cm, and σ = 7.5 cm. The ratio of the probability that the person is female to male is given by:

exp(-(x-*μ*f)2/2σ2) / exp(-(x-*μ*m)2/2σ2) = 0.647 / 0.923 = 0.700and so the probability that the person is female can be shown to be 0.412 or 41.2%.

What is the likelihood that a person over 190 cm is male?

In this case, one must compute the following integral using *μ*f and then *μ*m:

This yields *P*f(190) = 0.043 % and *Pm*(190) = 2.28%, which, under the assumption the total number of females and males is equal, yields.

*P*m(190) = 98.1%. (and *P*f(190) = 1.9%)